

Plane symmetric analogue of NUT space

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Abstract

In this article on the basis of a new definition of spacetime symmetry, which is in accordance with the symmetry of the curvature invariants, we investigate exact vacuum solutions of Einstein field equations corresponding to both static and stationary *plane symmetric* spacetimes using the concepts of the (1+3)-decomposition or *threading* formalism. Demanding the presence of a plane symmetric gravitomagnetic field we find a family of two parameter (m and ℓ) solutions, every member of which being the plane symmetric analogue of NUT space.

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I. INTRODUCTION

The (1 + 3)-decomposition (threading) of a spacetime by a congruence of timelike curves (observer worldlines) leads to the following splitting of the spacetime interval element [1];

$$ds^2 = dT^2 - dL^2 \quad (1)$$

where dL and dT are defined to be *the invariant spatial and temporal length elements* of two nearby events respectively. They are constructed from the normalized tangent vector $u^a = \frac{\xi^a}{|\xi|}$ to the timelike curves in the following way ¹;

$$dL^2 = h_{ab}dx^a dx^b \quad (2)$$

$$dT = u_a dx^a \quad (3)$$

where

$$h_{ab} = -g_{ab} + u_a u_b \quad (4)$$

is called the *projection tensor*. Taking $h \equiv |\xi|^2$ and $A_a \equiv -\frac{\xi_a}{|\xi|^2}$, equations (1) and (4) can also be written in the following alternative forms;

$$ds^2 = h(A_a dx^a)^2 - h_{ab}dx^a dx^b \quad (5)$$

$$h_{ab} = -g_{ab} + h A_a A_b$$

Using the preferred coordinate system in which the timelike curves are parameterized by the coordinate time x^0 of the comoving observers²;

$$\xi^a \doteq (1, 0, 0, 0) \quad \& \quad A_a \doteq (-1, -\frac{g_{0\alpha}}{g_{00}}) \quad (6)$$

¹Note that Latin indices run from 0 to 4 while the Greek ones from 1 to 3 and throughout we use gravitational units where $c=G=1$.

²Hereafter equations written in this preferred coordinate system are denoted by the sign \doteq .

then above spatial and spacetime distance elements will take the following forms [2];

$$dL^2 \doteq dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta \quad (7)$$

$$ds^2 = e^{2\nu} (dx^0 - A_\alpha dx^\alpha)^2 - dl^2 \quad (8)$$

where

$$e^{2\nu} \equiv g_{00} \quad A_\alpha \doteq -g_{0\alpha}/g_{00} \quad (9)$$

and

$$\gamma_{\alpha\beta} = (-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}). \quad (10)$$

Introducing *gravitoelectric*³ and *gravitomagnetic* fields [3];

$$\mathbf{E}_g = -\nabla\nu \quad (11)$$

$$\mathbf{B}_g = \text{curl}\mathbf{A} \quad (12)$$

one can write the vacuum Einstein equations for stationary spacetimes⁴ in the following quasi-Maxwell form [3];

$$\text{div}\mathbf{B}_g = 0 \quad (13)$$

$$\text{Curl}\mathbf{E}_g = 0 \quad (14)$$

$$\text{div}\mathbf{E}_g = -\frac{1}{2}e^{2\nu}B_g^2 + E_g^2 \quad (15)$$

$$\text{Curl}(e^\nu\mathbf{B}_g) = 2\mathbf{E}_g \times e^\nu\mathbf{B}_g \quad (16)$$

$$P^{\alpha\beta} = E_g^{\alpha;\beta} + e^{2\nu}(B_g^\alpha B_g^\beta - B_g^2 \gamma^{\alpha\beta}) + E_g^\alpha E_g^\beta \quad (17)$$

where $P^{\alpha\beta}$ is the three dimensional Ricci tensor constructed from the metric $\gamma^{\alpha\beta}$. One can show that the gravitoelectromagnetic fields can be written in the following covariant forms;

$$(E_g)_b = -\frac{1}{2} \frac{(\xi_a \xi^a)_{;b}}{|\xi|^2} = -\frac{1}{2} \frac{h_{;b}}{h} \quad (18)$$

³For Reviews on the subject of Gravitoelectromagnetism see references [3] and [4].

⁴For stationary spacetimes the preferred coordinate system is the one adapted to the congruence of its timelike killing vector, $\xi_t = \partial_t$.

$$B_g^b = -\frac{1}{2}|\xi|\xi^a\varepsilon_a^{bcd}\left[\left(\frac{\xi_d}{|\xi|^2}\right)_{;c} - \left(\frac{\xi_c}{|\xi|^2}\right)_{;d}\right] = \frac{1}{2}\sqrt{h}\xi^a\varepsilon_a^{bcd}(A_{d;c} - A_{c;d}) \quad (19)$$

One can also show that the gravitational Lorentz force on a test particle due to the spacetime curvature is given by [2,3];

$$\mathbf{f} = \frac{m_0}{\sqrt{1-v^2}}[\mathbf{E}_g + \mathbf{v} \times (e^\nu \mathbf{B}_g)] \quad (20)$$

This force deviates test particles from geodesics of the 'space' and make them follow the geodesics of spacetime [2,3]. It should be noted that the above (gravitoelectromagnetic) vector fields \mathbf{E}_g , \mathbf{B}_g , \mathbf{A} and the 3-dimensional tensor field $P^{\alpha\beta}$ are defined on a 3-dimensional manifold Σ (whose metric is $\gamma_{\alpha\beta}$). For a general spacetime manifold \mathcal{M} , Σ is defined as the factor space $\frac{\mathcal{M}}{\sim}$, where \sim is the equivalence relation which brings all the points on each threading curve under one class. In the case of stationary spacetimes this equivalence relation is given by the one-dimensional group G_1 of transformations generated by its timelike Killing vector ξ_t [5]. Physically this manifold, with the above spatial distance element dl , can be recognized as *the observed (3-dimensional) space of events* in the sense that dl is the spatial distances between events as measured by the observers on the timelike curves of the congruence [3,5]. From the mathematical point of view one should note that this is an abstract 3-manifold whose Riemannian metric $\gamma_{\alpha\beta}$ does not correspond to any hypersurface as its natural habitat!. This is in contrast to the (3+1)-decomposition (or slicing) of spacetimes [6] where the 3-dimensional metric lives on the spacelike hypersurfaces. In fact it can be shown that Σ has a one-parameter family of Riemmanian metrics and in this way one can describe spacetimes in the language of *parametric manifolds* [6,7].

II. WHAT DO WE MEAN BY A PLANE SYMMETRIC SPACETIME?

Plane symmetric static spacetimes have been found long time ago by Taub [8]. But using the concepts of 'threading' approach to the spacetime decomposition one can show that finding a metric with a certain symmetry is a different problem from finding a spacetime with the same symmetry. A known example is the NUT spacetime [9], although the

metric itself dose not have spherical symmetry but the spacetime really does i.e. all the curvature invariants of the spacetime are spherically symmetric [10,3]. The same property have been shown for the cylindrical symmetry through the so called *cylindrical analogue of NUT space* [11]. This difference is due to the fact the physical symmetry of a spacetime is the one associated with its spatial metric $\gamma_{\alpha\beta}$ and its gravitoelectromagnetic fields \mathbf{E}_g and \mathbf{B}_g and therefore it may or may not be followed by the gravitomagnetic potential \mathbf{A} [3] (See section III below for a general derivation). As a matter of fact under the simultaneous transformations;

$$A_\alpha \rightarrow A'_\alpha = A_\alpha + \phi_{,\alpha}(x^\beta) \quad (21)$$

$$g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = g_{\alpha\beta} + g_{00}(A_\alpha\phi_{,\beta} + A_\beta\phi_{,\alpha} + (\phi_{,\alpha})(\phi_{,\beta})) = g_{\alpha\beta} + g_{00}(A'_\alpha - A_\alpha)(A'_\beta + A_\beta) \quad (22)$$

the spatial metric γ_{ab} and g_{00} are unchanged i.e. metric will be physically unchanged, except for the changes in its time zero i.e. [2];

$$x^0 \rightarrow x^0 - \phi(x^\alpha) \quad (23)$$

This is not only true for stationary spacetimes but also for non-stationary spaces and it can be taken as a hint for studying the non-stationary spacetimes in the context of gravitoelectromagnetism [13].

III. PHYSICAL SYMMETRY OF A SPACETIME : GAUGED MOTION

The well known formulation of spacetime symmetries is based on the concepts of Killing vectors and the Killing equation (or Killing motion);

$$\mathcal{L}_\xi g_{ab} = 0 \quad (24)$$

which is obtained by the requirement of the invariance of the spacetime line element under an infinitesimal motion along a vector ξ^a ;

$$x^a \rightarrow x^a + \delta\lambda \xi^a \quad \delta\lambda \ll 1 \quad (25)$$

Here we use the same mathematical formulation but apply it not to the spacetime line element ds^2 but to the spatial and temporal line elements dT^2 and dL^2 of (1). This should be done in such a way that the freedom in choosing the time zero to be incorporated in the definition of (*physical*) symmetry. To do so we start from the spacetime line element in form (5);

$$ds^2 = h(A_a dx^a)^2 - h_{ab} dx^a dx^b$$

and we ensure the physical invariance under the symmetry operation (25) by the following requirements;

(a)- $\delta(dL^2) = 0$ from which we have;

$$\mathcal{L}_\xi h_{ab} = 0 \tag{26}$$

(b)- $\delta h = 0$ by which we get;

$$\mathcal{L}_\xi h = 0$$

which in turn by (18) reduces to;

$$\xi^a (E_g)_a = 0$$

and now due to the fact that $(E_g)_{a,b} = (E_g)_{b,a}$ we have,

$$\mathcal{L}_\xi (E_g)_a = 0 \tag{27}$$

(c)- finally to incorporate the freedom in choosing time zero we need

$$\delta(A_a dx^a) = \delta\lambda d\phi \tag{27a}$$

where ϕ is an arbitrary scalar function such that $\phi(x^a) \doteq \phi(x^\alpha)$, this is so because under the transformation

$$dx^0 \quad \rightarrow \quad dx'^0 = dx^0 - \delta\lambda d\phi(x^a)$$

we have

$$A_a dx^a \quad \rightarrow \quad A_a dx^a + \delta\lambda d\phi \doteq -dx^0 + A_\alpha dx^\alpha + \delta\lambda d\phi$$

on the other hand under (25)

$$\delta(A_a dx^a) = \delta\lambda dx^a \mathcal{L}_\xi A_a \quad (27b)$$

Now comparing equations (27a) and (27b) one gets;

$$\mathcal{L}_\xi A_a = \phi_{;a}$$

Using the above relation for the Lie derivative of the gravitomagnetic potential and the definition of $F_{ab} = A_{b;a} - A_{a;b}$ one could easily show that;

$$\mathcal{L}_\xi F_{ab} = 0 \quad (28)$$

where F_{ab} , in the preferred coordinate system, has the following form;

$$F_{ab} \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{B}^3 & -\hat{B}^2 \\ 0 & -\hat{B}^3 & 0 & \hat{B}^1 \\ 0 & \hat{B}^2 & -\hat{B}^1 & 0 \end{pmatrix}$$

or in a more compact, 3-dimensional notation;

$$\hat{B}^\alpha = \sqrt{\gamma} B^\alpha = \frac{1}{2} \sqrt{\gamma} \varepsilon^{\alpha\beta\gamma} F_{\beta\gamma}$$

One should note that in the above discussion the appearance of E_a and F_{ab} (and consequently the components B^α) is completely independent of their introduction through the Einstein field equations in the quasi-Maxwell form. Furthermore we have shown that the physical symmetry of a spacetime is the one which is respected by its gravitoelectromagnetic fields and the spatial metric h_{ab} .

A. Gauged motion

As a consequence of relation (4) and equations (26)-(28), we obtain,

$$\mathcal{L}_\xi g_{ab} = h\phi_{;(a} A_{b)} \quad (29)$$

This introduces a new generalization of the usual Killing motion (based on the Killing equation (24)), which we call a *gauged Killing motion* with the corresponding *gauged Killing vector*.⁵ It can be shown [13] that this motion is in accordance with the symmetry of the curvature invariants of a spacetime and therefore with the spacetime symmetry itself as it has already been shown for the spherical [3,10] and cylindrical [11] cases. In the next section we will apply the above formalism to the special case of plane symmetric spacetimes.

IV. PLANE SYMMETRIC SPACETIMES

Applying the above ideas to the case of plane symmetry we use the Cartesian coordinates x, y, z and choose the z direction as the distinguished one for describing the plane symmetry. Now working in the preferred coordinate system in which the Killing vectors are given by

$$\begin{aligned}\xi_{(0)}^a &= (1, 0, 0, 0) & \xi_{(1)}^a &= (0, 1, 0, 0) \\ \xi_{(2)}^a &= (0, 0, 1, 0) & \xi_{(3)}^a &= (0, -y, x, 0)\end{aligned}$$

and using equations (26)-(28) one can show that for a plane symmetric spacetime the following conditions are required to be satisfied;

- (a)- from equation (27) and for $\xi_{(1)}^a$ and $\xi_{(2)}^a$ we find $\nu_{,x} = \nu_{,y} = 0$ i.e. g_{00} is only z -dependent. By this condition, on each (x, y) plane, the rates of the clocks at different points are equal and the gravitoelectric field is a constant vector with the only component $E_{g3}(z)$.
- (b)- from equation (26) and for $\xi_{(1)}^a$ and $\xi_{(2)}^a$ one finds, $\gamma_{\alpha\beta,x} = \gamma_{\alpha\beta,y} = 0$ i.e. components of the spatial metric are only z dependent.
- (c)- from equation (26) and for $\xi_{(3)}^a$ we find that on each (x, y) plane, distance elements along the x and y directions are equal and hence we need $\gamma_{xx} = \gamma_{yy}$.
- (d)- from equation (28) and for $\xi_{(1)}^a$, $\xi_{(2)}^a$ and $\xi_{(3)}^a$ one can show that the gravitomagnetic

⁵One should note that there are other generalization of equation (24) such as the *conformal* and *homothetic* Killing motions [12].

field should only be z -dependent and that $B_x = B_y = 0$.

Applying the above conditions and the existence of a timelike Killing vector $\xi_{(0)}^a$, we obtain the following general form for a plane symmetric spacetime;

$$ds^2 = e^{2\nu(z)}(dx^0 - A_\alpha dx^\alpha)^2 - \gamma_{\alpha\beta}(z)dx^\alpha dx^\beta$$

On the other hand using the definition of divergence in γ -space and the fact that $B_g^\alpha = (0, 0, B_g^3(z))$, from equation (2) we have;

$$B_g^3 = \ell/\sqrt{\gamma} \quad (30)$$

where $\gamma = \det\gamma_{\alpha\beta}$ and ℓ is a constant indicating the gravitomagnetic field strength. Of course there are different choices for the vector field \mathbf{A} giving rise to the same \mathbf{B}_g field but as we have discussed earlier they are related through gauge transformations induced by a shift in the time zero. The simplest choices which will do the job are $A_\alpha = (0, \ell x, 0)$ and $A_\alpha = (-\ell y, 0, 0)$. Choosing the first form for the vector potential the metric of a stationary plane symmetric spacetime with diagonal γ matrix will find the following general form;

$$ds^2 = e^{2\nu(z)}(dx^0 - \ell x dy)^2 - e^{\lambda_\alpha(z)}(dx^\alpha)^2 \quad (31)$$

where $\gamma_{\alpha\alpha} = e^{\lambda_\alpha(z)}$. For the moment we forget the condition $\gamma_{11} = \gamma_{22}$ to obtain a more general solution. Using the form (9) the vacuum equations $R_{ab} = 0$ or their equivalent quasi-Maxwell form lead to the following equations;

$$2\nu'' + \nu'(2\nu' + \lambda'_1 + \lambda'_2 - \lambda'_3) + \ell^2 e^{2\nu+\lambda_3-\lambda_1-\lambda_2} = 0 \quad (32)$$

$$-2\lambda''_1 - \lambda'_1(2\nu' + \lambda'_1 + \lambda'_2 - \lambda'_3) + 2\ell^2 e^{2\nu+\lambda_3-\lambda_1-\lambda_2} = 0 \quad (33)$$

$$-2\lambda''_2 - \lambda'_2(-2\nu' + \lambda'_1 + \lambda'_2 - \lambda'_3) + 2\ell^2 e^{2\nu+\lambda_3-\lambda_1-\lambda_2} = 0 \quad (34)$$

$$4\nu'' + 4(\nu')^2 - \lambda'_3(2\nu' + \lambda'_1 + \lambda'_2) + 2\lambda''_1 + 2\lambda''_2 + (\lambda'_1)^2 + (\lambda'_2)^2 = 0 \quad (35)$$

Before looking for the general solution of the above equations we solve them in two special cases.

V. THE STATIC CASE

In this case we set $\ell = 0$ and there are no cross terms in the metric and consequently $\mathbf{B}_g = 0$. The resulted equations are easy to solve and the general solution is given by;

$$\begin{aligned}
 g_{00} &= e^{2\nu} \\
 g_{11} &= -k_1 e^{c\nu} \\
 g_{22} &= -k_2 e^{\frac{-2c}{c+2}\nu} \\
 g_{33} &= -k_3 \nu'^2 e^{(2+\frac{c^2}{c+2})\nu}
 \end{aligned} \tag{36}$$

where k_α s and c are constants to be determined. We set $k_1 = k_2 = 1$ as they can be absorbed in the coordinates x_1 and x_2 respectively. Applying the third condition of plane symmetry, i.e. $\gamma_{11} = \gamma_{22}$, to the above metric we get $c = -4$ and so the metric will take the following form;

$$\begin{aligned}
 g_{00} &= e^{2\nu} \\
 g_{11} &= g_{22} = -e^{-4\nu} \\
 g_{33} &= -k \nu'^2 e^{-6\nu}
 \end{aligned} \tag{37}$$

where $k = k_3 > 0$ is the only constant left to be determined. One should note that this is the same metric found by Taub in his search for a plane symmetric metric. His metric is given by [8];

$$ds^2 = \frac{1}{\sqrt{1+\kappa Z}}(dt^2 - dZ^2) - (1+\kappa Z)(dx^2 + dy^2) \tag{38}$$

where $\kappa = \text{constant}$. The two forms (38) and (37) can be transformed into one another by the following transformation;

$$\frac{1}{\sqrt{1+\kappa Z}} = e^{2\nu(z)}$$

with $k = \frac{\kappa^2}{4}$.

A. A candidate metric for the spacetime of an infinitely large massive wall

Now we would like to find a candidate for the spacetime of an infinitely large massive wall, whose metric is expected to belong to the above general plane symmetric static family. But before looking for a physical potential $\nu(z)$, using the general form (37), we show that the asymptotic flatness of such a spacetime is in contradiction with its gravity field being attractive. This is so because its Riemann invariant is found to be

$$R^{abcd}R_{abcd} = \frac{192}{k^2}e^{12\nu(z)}$$

therefore to get asymptotic flatness we need $\nu(z) \rightarrow -\infty$ as $z \rightarrow \infty$. On the other hand an attractive gravitoelectric field for all $z > 0$ requires a monotonically increasing $\nu(z)$ which is obviously inconsistent with the previous requirement. This result was not unexpected as the plane symmetry requirement naturally brings in infinitely extended sources. An infinitely large massive wall in Newtonian gravity produces the Newtonian gravitoelectric field $\mathbf{E}_g = -\frac{m}{2}\hat{z}$ for $z \geq 0$, where m is the *mass density* on the wall (the case for $z \leq 0$ is very similar). To obtain the candidate metric of such a wall we need to introduce a gravitoelectric potential $\nu(z)$ along with a proper choice of the constant k such that the resulted metric fulfills the following two requirements; First of all, in the Newtonian limit, it should produce the gravitoelectric field $\mathbf{E}_g = -\frac{m}{2}\hat{z}$ and secondly, for $m = 0$, the metric should reduce to that of the flat spacetime.

One can easily see that the choices

$$e^{2\nu(z)} = z^{m^2}e^{mz} \quad \text{and} \quad k = 4/m^2 \quad (39)$$

will do the job and the metric will take the following form;

$$ds^2 = z^{m^2}e^{mz}dt^2 - (z^{-2m^2}e^{-2mz})(dx^2 + dy^2) - (z^{-3m^2}e^{-3mz})(1 + m/z)^2dz^2 \quad (40)$$

As it can easily be seen this metric is flat for $m = 0$ and its gravitoelectric field $E_g = -\frac{m}{2}(1 + m/z)$ is different from that in the Newtonian case but tends to it as $z \rightarrow \infty$ (though

the source is not confined within a limited region of space, it is localized in the z -direction). The spacetime is not asymptotically flat and the metric components have nearly the same behaviour at $z = 0$ and $z \rightarrow \infty$ as does Levi Civita's cylindrically symmetric metric [14] at $\rho = 0$ and $\rho \rightarrow \infty$.⁶

VI. THE STATIONARY CASE

The sharpest distinction between the stationary and the static cases is of course the appearance of the gravitomagnetic field (of course we need the cross term to be coordinate dependent), to which 'nonzero rest mass' particles respond with their velocities by the second term of the gravitoelectromagnetic force (20). Normally it is expected that such a field to be produced by mass currents but there are some interesting exceptions such as the NUT-type spacetimes whose gravitomagnetic could not be attributed to any kind of mass current in a consistent way. These are stationary solutions of Einstein field equations with two parameters, the mass parameter m and the so called NUT factor ℓ (also called magnetic mass) which is the source of the gravitomagnetic field [10,3]. It has already been shown that the famous *spherical* NUT spacetime and its *cylindrical analogue* are the empty space generalizations of the Schwartzchild and Levi-civita spacetimes respectively. In what follows we will find the plane symmetric analogue of NUT space and show that it is the empty space generalization of the metric (40) of the infinitely large massive wall. But before that we find another solution of equations (32)-(35) which is a special case of our final plane symmetric analogue of NUT metric and that is the one parameter ($\ell \neq 0$ and $m = 0$) metric corresponding to the spacetime of a distribution of magnetic masses (NUT parameter) over an infinite plane, what can be called the plane symmetric analogue of pure NUT space.

⁶We compare (40) with Levi Civita's cylindrically symmetric metric because they are both expected to be the spacetimes of noncompact sources.

A. PLANE SYMMETRIC ANALOGUE OF PURE NUT SPACETIME

The following special choice;

$$\lambda_3 = \lambda_1 + \lambda_2 + 2\nu \quad (41)$$

cancels out a lot of terms in equations (32)-(35) and what is left can be easily solved. The only solution which reduces to that of the Minkowski space when $\ell \rightarrow 0$ is found to be;

$$\begin{aligned} \nu(z) &= -\frac{1}{2} \ln(\cosh(\ell z)) \\ \lambda_1 &= \ln(\cosh(\ell z)) + c\ell z \\ \lambda_2 &= \ln(\cosh(\ell z)) + \frac{1}{c}\ell z \\ \lambda_3 &= \ln(\cosh(\ell z)) + (c + \frac{1}{c})\ell z \end{aligned} \quad (42)$$

where c is turned up to be equal to 1 by the symmetry requirement $\gamma_{xx} = \gamma_{yy}$ and so the metric will take the following form;

$$\begin{aligned} g_{00} &= \frac{1}{\cosh(\ell z)} \\ g_{11} &= -\cosh(\ell z)e^{\ell z} \\ g_{22} &= -\cosh(\ell z)e^{\ell z} + \frac{\ell^2 x^2}{\cosh(\ell z)} \\ g_{33} &= -\cosh(\ell z)e^{2\ell z} \\ g_{01} &= -\frac{\ell x}{\cosh(\ell z)} \end{aligned} \quad (43)$$

This is a stationary spacetime in which the source of the spacetime curvature is not dependent on the mass but on another parameter, the gravitomagnetic field strength ℓ . This can also be seen by looking at its gravitomagnetic and gravitoelectric fields which are;

$$B_g = \ell \frac{e^{-2\ell z}}{\cosh^{3/2}(\ell z)} \quad (44)$$

$$E_g = \frac{\ell}{2} \tanh(\ell z) \quad (45)$$

One can compare (43-45) with the usual pure ($m = 0$) NUT space where we have a spherically symmetric stationary spacetime whose source is the NUT factor ℓ and not the mass. Next we find the promised general family of plane symmetric analogue of NUT spacetimes where (40) and (43) are its limiting cases for $\ell = 0$ and $m = 0$ respectively.

VII. PLANE SYMMETRIC ANALOGUE OF NUT SPACE

To find the general plane symmetric analogue of NUT space (i.e. $m \neq 0$, $\ell \neq 0$) we introduce the following convenient new variable;

$$2F(z) = \lambda_3 - \lambda_1 - \lambda_2 - 2\nu \quad (46)$$

into equations (32)-(35) which simplifies them and one can show that their general solution is given by;

$$\lambda_1 = -2\nu - 2c \operatorname{arctanh}(\sqrt{1 - \ell^2 d^2 e^{4\nu}}) + \ln(d_1) \quad (47)$$

$$\lambda_2 = -2\nu - \frac{1}{2c} \operatorname{arctanh}(\sqrt{1 - \ell^2 d^2 e^{4\nu}}) + \ln(d_2) \quad (48)$$

$$F(z) = \ln(\lambda'_1 + 2\nu') + \ln(d) \quad (49)$$

where coefficients d_1 , d_2 , d and c are constants to be determined using symmetries and physical arguments. First of all by the symmetry condition $\gamma_{xx} = \gamma_{yy}$, we find $c^2 = \frac{1}{4}$ and $d_1 = d_2$. Now the requirement that this solution reduces to that of the general static solution (equation (37)) for $\ell = 0$ and to that of the flat space for $m = 0$ and $\ell = 0$, leads us to the following choices of the constants in the above functions;

$$d_1 = d_2 = 2^{\frac{-m}{m+\ell}} \frac{\ell}{(m+\ell)^2} \quad (50)$$

$$c = -1/2 \quad d = \frac{1}{(m+\ell)^2} \quad (51)$$

Now we Choose a suitable potential $\nu(z)$ so that we could recover the candidate spacetime of an infinitely large massive wall when $\ell = 0$ and the plane symmetric analogue of pure NUT space (equation (43)) when $m = 0$. One can show that the following choice fulfills

these requirements and further we have a well-defined metric everywhere (for all values of z);

$$\nu(z) = \frac{1}{2} \left[\frac{N m z}{m z + N} + \ln \left(\frac{z^{m^2}}{\cosh(\ell z)} \right) \right] \quad (52)$$

where $N = \ln(\frac{m}{\ell} + 1)$. This concludes our general stationary plane symmetric analogue of NUT space which has two parameter m and l .

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